Eur. Phys. J. B **28**, 263–269 (2002)
DOI: 10.1140/epjb/e2002-00228-4

THE EUROPEAN PHYSICAL JOURNAL B

Effective magnetic properties of a composite material with circular conductive elements

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Received 19 April 2002
Published online 31 July 2002 –

Abstract. Effective magnetic properties of a composite meta-material consisting of periodically arranged circular conductive elements are studied theoretically. A general expression for the effective bulk permeability is obtained with mutual effects and lattice ordering being taken into account. The resonance frequency of the permeability is found to be strongly dependent on the size and shape of the unit cell. Frequency dispersion of the permeability is studied with special attention paid to the frequency range, where negative values of the permeability are possible. Corresponding recommendations for optimisation of the meta-materials with negative permeability are made. The results are confirmed by numerical simulations of the finite structure behaviour in an external magnetic field.

PACS. 41.20.Jb Electromagnetic wave propagation; radiowave propagation – 78.20.Bh Theory, models, and numerical simulation

1 Introduction

A novel class of artificial materials consisting of periodic arrays of small conductive elements appears to attract increasing attention of many physicists due to possible unusual electromagnetic features in the microwave frequency range [1]. For example, the possibility of obtaining negative permeability values makes such materials applicable for experiments dealing with negative refraction. The negative refraction phenomenon, *i.e.*, wave propagation in media with simultaneously negative permeability and permittivity [2,3], is practically unavailable in optics, as the permeability of all known media is close to unity at optical frequencies. Meta-materials provide, therefore, an exceptional opportunity for the practical manifestation of the negative refraction idea [4].

If the wavelength of the electromagnetic wave inside the medium is much larger than both the element size and the distances between neighbouring elements, the electromagnetic properties of such a material are well described by effective permeability and permittivity. In general, the possibility to construct the medium from structural units possessing predetermined controllable properties provides a rare chance to study how effective macroscopic parameters of the substance are built up. This makes the whole field very interesting also from a general physical point of view, as one can apply here the well developed apparatus of the electrodynamics of condensed matter [5,6], hoping to obtain results which are quantitatively even more accurate than in the case of the optical theory where the structural units, *i.e.*, atoms and molecules, are normally too complicated for a simple analytical treatment.

To obtain the effective response of the macroscopic meta-material from the known properties of the structural units and the periodical lattice parameters we develop a theory which has much in common with the theory of effective macroscopic dielectric response in crystals [7–9]. The general features of the latter can be summarized as:

– On the microscopic level mutual effects of the structural units are taken into account.
– The resulting microscopic non-locality is described by a characteristic length of response formation (local field formation length in local field theory). As a rule, it is of the order of several lattice constants.
– The final effective response in the bulk area of the material can be considered as local if the wavelength of the electromagnetic wave is larger than the response formation length.
– Near material borders, a subsurface transition region appears with properties different from those in the volume. The characteristic thickness of the subsurface area is of the order of the response formation length.

A prominent example of the simplest structural unit is a circular conductive element (loop), possessing self-inductance, implemented capacitance and ohmic resistive losses. Corresponding meta-materials are already available for experimental research [4,10]. Nevertheless, the theoretical description of the effective magnetic properties of

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such substances leaves much to be desired, as mutual effects were not properly taken into account. In particular, the resonance frequency of the material permeability was concluded to be equal to that of a single structural unit. As known from optical response theory, the excitations in crystals are of collective nature and mutual interactions between atoms affect strongly the position of the resonance [11]. As we shall show, the same happens also in the case of meta-material – the resonance frequency obtained below depends markedly on the lattice parameters. At frequencies above the material resonance, the permeability is negative. Performed analysis of the frequency dispersion of the permeability allows optimisation of the arrangement of the structural elements so that one can choose the most favourable lattice type and lattice constants to achieve, for example, a broader frequency range of the negative permeability.

Numerical simulation of the electromagnetic response of a finite sample of the meta-material under the action of an external magnetic field proved completely the assumptions and results of the macroscopic approach. This fact does not only ensure the general principles but also allows us to expect that one can successfully use the numerical technique in cases when the structural units are not appropriate for analytical treatment. The most fruitful and, perhaps, the only possible way would be to solve numerically the field problem in a structure containing about thousand elements. Numerical averaging yields then the adequate macroscopic characteristics in a wide frequency range.

2 Averaging procedure and effective permeability

We suppose the circular loops to be arranged so that the loop planes are parallel and the centres of the loops form a kind of regular lattice. One should certainly keep in mind that the microscopic dimensions of the artificial substance are several orders of magnitude larger than those in crystals. The condition that the electromagnetic wavelength should be much larger than microscopic structural details is automatically fulfilled in crystals for the optical range. In the case of meta-materials, the analogous condition shifts the appropriate frequency range to microwaves with wavelengths of the order of several centimetres. Another difference, which is actually not so critical for the approach itself but appears during the interpretation of the results, is that we focus on how the loops affect magnetic field, while most effects in crystal optics are concerned with electric field. Due to the electric field – magnetic field symmetry of Maxwell's equations it is rather easy to find corresponding analogies between crystals and meta-materials. Nevertheless, the physical background is quite different.

As in the local response theory [5,7], we postulate that the response is formed at distances much smaller than the wavelength, *i.e.*, we can neglect retardation while considering loop interactions. This quasi-static limit allows us to separate magnetic effects from electric ones so that only the magnetic field affects the magnetization of the medium, defining the permeability. Electric fields are coupled by the polarization, define the permittivity and do not interfere with the magnetic properties at this stage. The resulting magnetic permeability is also not affected by the wavelength so that the latter can be treated as infinite. Then one can neglect the inhomogeneity of the magnetization and of the averaged fields and the problem reduces to the behaviour of loop arrays in the external homogeneous oscillating magnetic field.

Let all the loops lie in parallel planes normal to the z-axis. The structure is supposed to be infinite and the centres of the loops are located at the points $\mathbf{r}_n$. These points are assumed to form a regular spatial lattice so that each loop has the same surrounding. We suppose the material of the structure elements to be non-magnetic so that the magnetization is only due to the currents induced in the loops. Though in general one should consider the full tensor of the permeability, from the chosen geometry it is obvious that the magnetization has only z-component, *i.e.*, only μ_{zz} differs from unity. Therefore, only z-component of the magnetic field is important, and the problem becomes scalar.

Supposing the time dependence of fields and currents to have the oscillating form $\mathrm{e}^{-\mathrm{i}\omega t}$, one can write the electro-motive force $\mathcal{E}_n$ in the nth loop, induced by the external field $\mathbf{H}_0$, as

$$\mathcal{E}_n = \mathrm{i}\omega\mu_0\pi r_0^2 H_{0z}, \tag{1}$$

where r_0 is the radius of the loop. Using the multi-impedance matrix allows to write

$$\mathcal{E}_n = ZI_n + \sum_{n'\neq n} Z_{nn'} I_{n'}, \tag{2}$$

where I_n is the current induced in the nth loop, Z is the self-inductance and $Z_{nn'}$ is the mutual inductance between the loops n and n'.

The magnetic properties of each loop are defined by the same self-impedance Z so that we can write

$$Z = -\mathrm{i}\omega L + \frac{\mathrm{i}}{\omega C} + R. \tag{3}$$

Here we can treat the self-inductance L, the capacitance C, and the resistance R as predefined. Below we estimate them for a practically interesting case.

Obviously, due to the homogeneity of the external field and infinity of the medium, all the loops are in the same situation. Thus, all the currents $I_n = I$ are equal. The magnetic moment of each loop equals to $\pi r_0^2 I$.

The averaged media magnetization defined as the magnetic moment density is given by

$$M_z = n_0\pi r_0^2 I, \tag{4}$$

where the volume concentration of loops n_0 is introduced. Combining equations (1, 2, 4) together allows to obtain

$$\left(Z + \sum_{n'\neq n} Z_{nn'}\right) M_z = \mathrm{i}\omega\pi^2 r_0^4 n_0 \mu_0 H_{0z}. \tag{5}$$

The total microscopic magnetic field at the point $\mathbf{r}$ is given by the sum of the external field and the contribution of the loops:

$$H_{\mathrm{mic}\,z}(\mathbf{r}) = H_{0z} + \sum_n H_l(\mathbf{r} - \mathbf{r}_n), \tag{6}$$

where the function $H_l(\mathbf{r}')$ is defined as the value of the z-component of the magnetic field induced by the loop, located at the coordinate origin, at the point $\mathbf{r}'$. According to the Biot-Savart's law $H_l(\mathbf{r}')$ can be presented as an integral along the loop contour

$$H_l(\mathbf{r}') = \frac{I}{4\pi}\int \frac{[\mathrm{d}\,\mathbf{l} \times (\mathbf{r}' - \mathbf{s})]_z}{|\mathbf{r}' - \mathbf{s}|^3}, \tag{7}$$

where the vector $\mathbf{s}$ is the radius vector of that point of the loop, where $\mathrm{d}\,\mathbf{l}$ is taken.

Since all the unit cells are identical, the field distribution is the same in all the cells. Therefore, the macroscopic averaging can be performed over the volume $V_m = n_0^{-1}$ of one unit cell with any number m. The averaged value of the microscopic magnetic field (6) yields the macroscopic induction

$$B_z = \mu_0 \langle H_{\mathrm{mic}\,z}\rangle = \mu_0 H_{0z} + \frac{\mu_0}{V_m}\sum_n \int\limits_{V_m} \mathrm{d}\mathbf{r}\, H_l(\mathbf{r} - \mathbf{r}_n). \tag{8}$$

The radius vector $(\mathbf{r} - \mathbf{r}_n)$ passes all the cells with centres at $(\mathbf{r}_m - \mathbf{r}_n)$, where m takes all possible values. The summation over all n in (8) provides the result which is independent of the particular number m so that we can write

$$\begin{aligned} B_z &= \mu_0 H_{0z} + n_0\mu_0 \sum_{n'} \int\limits_{V_{n'}} \mathrm{d}\mathbf{r} H_l(\mathbf{r}) \\ &= \mu_0 H_{0z} + n_0\mu_0 \int\limits_{V} \mathrm{d}\mathbf{r} H_l(\mathbf{r}). \end{aligned} \tag{9}$$

The integration in the last term is to be performed over the large macroscopic volume V of the whole medium. We take the latter as the limit of a large sphere S centered in at the coordinate origin with the radius r_s going to infinity.

Using the relation

$$\lim_{r_s\to\infty} \int \mathrm{d}\mathbf{r}\frac{\mathbf{r}' - \mathbf{s}}{|\mathbf{r}' - \mathbf{s}|^3} = -\frac{4\pi}{3}\mathbf{s}, \tag{10}$$

it is easy to obtain

$$\int \mathrm{d}\mathbf{r} H_l(\mathbf{r}) = \frac{2}{3}\pi r_0^2 I = \frac{2}{3}\frac{M_z}{n_0}, \tag{11}$$

which enables us to conclude that generally

$$B_z = \mu_0\left(H_{0z} + \frac{2}{3}M_z\right). \tag{12}$$

The definitions $B_z = \mu_0\mu_{zz}H_z$ and $B_z = \mu_0(H_z + M_z)$ allow us to express the effective permeability μ_{zz} as

$$\mu_{zz} = \frac{B_z}{B_z - \mu_0 M_z}\cdot \tag{13}$$

Using equations (5) and (12) allows to rewrite it as

$$\mu_{zz} = \frac{Z + \sum\limits_{n'\neq n} Z_{nn'} + \frac{2}{3}\mathrm{i}\omega\pi^2 r_0^4 n_0\mu_0}{Z + \sum\limits_{n'\neq n} Z_{nn'} - \frac{1}{3}\mathrm{i}\omega\pi^2 r_0^4 n_0\mu_0}\cdot \tag{14}$$

The mutual inductance of the loops in the limit of infinitely thin wire is given by the double integral along the loops:

$$Z_{nn'} = \mathrm{i}\omega\frac{\mu_0}{4\pi}\iint \frac{(\mathrm{d}\,\mathbf{l}_n \cdot \mathrm{d}\,\mathbf{l}_{n'})}{|\mathbf{s}_n - \mathbf{s}_{n'}|}\cdot \tag{15}$$

For two circular loops with the centres at points $\mathbf{r}_n$ and $\mathbf{r}_{n'}$ this gives

$$Z_{nn'} = \mathrm{i}\omega\mu_0 r_0 J(\mathbf{r}_n - \mathbf{r}_{n'}) \tag{16}$$

where we use the dimensionless function

$$J(\mathbf{r}) = \int\limits_0^{2\pi}\int\limits_0^{2\pi} \frac{r_0\cos(\varphi_1-\varphi_2)\mathrm{d}\varphi_1\mathrm{d}\varphi_2}{4\pi\sqrt{\rho^2+z^2+2r_0^2(1-\cos(\varphi_1-\varphi_2))+2\rho r_0(\cos\varphi_2-\cos\varphi_1)}}, \tag{17}$$

where ρ and z are the components of the vector $\mathbf{r}$ in cylindric coordinates. Now we can rewrite the expression for the effective μ_{zz} as

$$\mu_{zz} = \frac{\dfrac{\mathrm{i}Z}{\omega\mu_0 r_0} + \Sigma - \dfrac{2}{3}\pi^2 r_0^3 n_0}{\dfrac{\mathrm{i}Z}{\omega\mu_0 r_0} + \Sigma + \dfrac{1}{3}\pi^2 r_0^3 n_0}, \tag{18}$$

where the dimensionless parameter $\Sigma = \sum\limits_{n'\neq n} J(\mathbf{r}_n - \mathbf{r}_{n'})$ depends only on the lattice type and the values of the lattice constants.

One can see that the relation (18) is affected by the lattice order *via* the sum Σ only. This summation is performed over all the loops, *i.e.*, over the macroscopic volume. This volume should be the same as for the averaging procedure, and we use the same spherical limit. Actually, it is necessary to perform the summation only over a finite and relatively small number of loops, which are located in the volume near the nth one. Further increase in the radius r_s does not influence the summation result. For a good numerical accuracy of a few percent it is sufficient to set r_s to be only six times larger than the lattice constant. This satisfactory value of the distance r_s can be considered as a characteristic length of the local response L_{resp}. Although for different lattice types and various lattice constants L_{resp} differs in magnitude, it is, as a rule, of the order of several inter-loop distances. The length L_{resp} is the parameter the wavelengths and sample dimensions should be compared with to make the macroscopic effective response approach valid.

3 Frequency dispersion of the permeability

The frequency dependence of the effective permeability given by (18) is controlled by the loop self-impedance. The general expression (3) is applicable for various conductive elements, provided that the current cross-section is much smaller than the element size. For more complicated structural units like split ring resonators [4] the self-impedance ceases precise analytical treatment, but can be easily determined experimentally by studying the response of a single element. For the illustrative analysis below we consider a thin loop made of wire with the radius l ($l \ll r_0$). The self-inductance of such a loop equals [5]

$$L = \mu_0 r_0 \left(\ln \frac{8r_0}{l} - 2 \right) + \frac{\mu_0 r_0}{4} \cdot \quad (19)$$

For a given inductance, we choose the value of the implemented capacitance C which is necessary for providing appropriate resonance frequency of a single loop. To be consistent with the available experimental information [4], we set the latter equal to 3 GHz. The corresponding value of the cyclic resonance frequency of a single loop,

$$\omega_0 = \sqrt{LC}, \quad (20)$$

equals $6\pi \times 10^9$ rad/s. Appropriate capacitance value could be obtained, for instance, by a small slit in the loop contour, so that

$$C = \varepsilon_0 \pi l^2 d^{-1},$$

where the slit width d should be of the order of microns for the loop radius $r_0 = 2$ mm and wire radius $l = 0.1$ mm. Actually, one can make capacitance plates larger than the wire cross section which will allow to increase the slit width.

To account for resistive losses in the wire one should consider skin effect, as the frequencies are rather high. The skin depth in a material with the characteristic conductivity of copper ($\sigma = 0.65 \times 10^8$ S/m),

$$\delta = \sqrt{\frac{1}{2}\sigma\omega\mu_0}, \quad (21)$$

appears to be several orders of magnitude smaller than the wire radius. Thus only thin subsurface layer of the wire is really active, and the resistance equals

$$R = r_0 (\sigma l \delta)^{-1}. \quad (22)$$

The corresponding quality factor of the loop is then of the order of thousand, which is in good agreement with the experimental data [4]. We use the physical parameters mentioned above for the illustrative calculations described below.

For the analysis of frequency dispersion, it is convenient to present the effective permeability (18) in the general resonance form:

$$\mu_{zz}(\omega) = 1 - \frac{A\omega^2}{\omega^2 - \omega_r^2 + \mathrm{i}\Gamma\omega}, \quad (23)$$

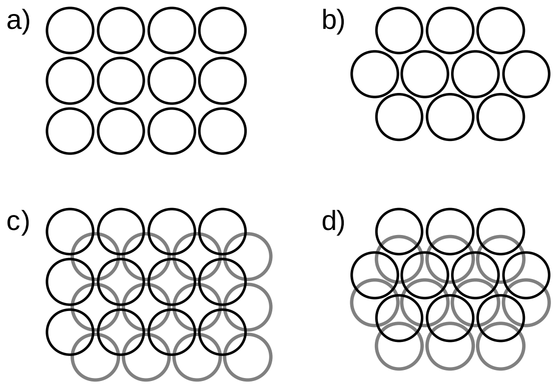

Fig. 1. Four types of lattices studied. Tetragonal (a), hexagonal (b), shifted tetragonal (c) and shifted hexagonal (d) lattices are shown as seen from the top of the multi-layered structure. In (c) and (d) cases two neighbouring layers are shown in different hue (black and grey circles, respectively).

where the resonance frequency of the medium is

$$\omega_r = \omega_0 \left(1 + \mu_0 r_0 \Sigma L^{-1} + \frac{\pi^2}{3} \mu_0 r_0^4 n_0 L^{-1} \right)^{1/2}, \quad (24)$$

the resonance amplitude is

$$A = \pi^2 \mu_0 r_0^4 n_0 L^{-1} \frac{\omega_r^2}{\omega_0^2}, \quad (25)$$

and damping is provided by

$$\Gamma = R L^{-1} \frac{\omega_r^2}{\omega_0^2} \cdot \quad (26)$$

Relation (24) demonstrates that the resonance of the medium can be sufficiently shifted from the single loop resonance due to the mutual interaction. This shift increases as the volume concentration of loops is increased. The exact values appear to depend, *via* the sum Σ, on the type of the regular spatial ordering. The reported lattices are of the layered type [4,10]. It appears to be technically simpler to pack first the loops in some regular order on flat surfaces and afterwards place these planes in a regular pattern to form the sample. With this technique, the simplest case seems to be the tetragonal lattice, where the loops form a square structure in the planes and all the planes are put just one above another so that columns of loops are formed. A view from the top of the medium is shown in Figure 1a. Packing within a plane can have another symmetry: the hexagonal one is shown in Figure 1b. It is also possible to shift the loops in neighbouring layers with respect to each other. We can call this a "shifted lattice". Extreme shifting is achieved when the centre of a loop is located equidistantly from the loop centres in the nearest layer. Such shifted tetragonal and hexagonal lattices are shown in Figures 1c and d, respectively (two neighbouring layers are shown).

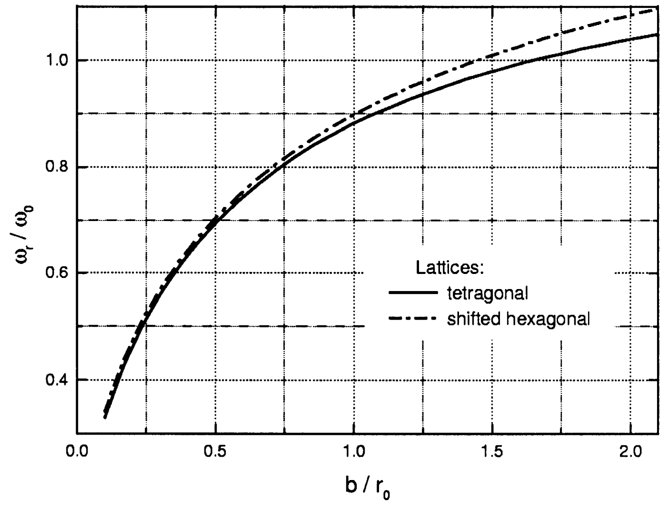

Fig. 2. Relative resonance frequency shift as a function of the ratio of the interlayer distance to the loop radius.

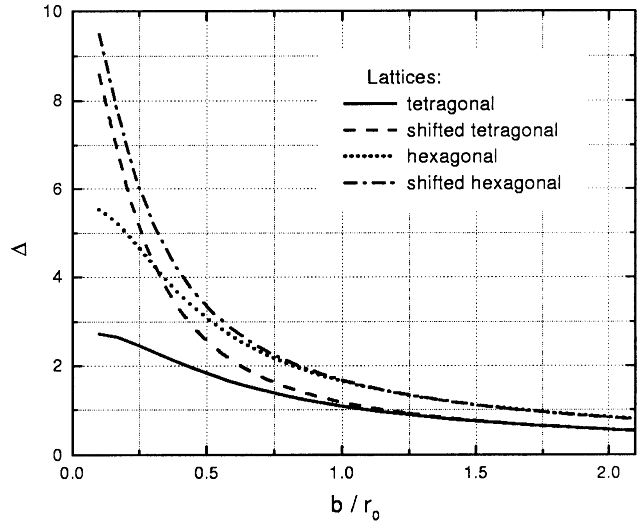

Fig. 3. Relative broadness Δ of the negative permeability frequency range as a function of the ratio of the interlayer distance b to the loop radius r_0.

Obviously, to enhance the mutual effects one should increase the volume concentration of loops. The distance a between the loop centres within a plane is limited by two loop radiuses; below we accept $a = 2.1\,r_0$. The interlayer distance b could be technically limited, we assume the value $b = 0.1\,r_0$ to be achievable.

The calculated resonance frequency dependencies on the lattice constant b for different lattices are shown in Figure 2. They depend only slightly on the lattice type, and we show the two most deviating curves, for tetragonal and shifted hexagonal arrangements. It is clear that one can drastically decrease the resonance frequency increasing packing density of the conductive elements.

At frequencies larger than ω_r the real part of the permeability (23) is negative. It grows with further increase in frequency and becomes positive at a frequency $\tilde{\omega}$, which can be found from the condition $\mu_{zz}(\tilde{\omega}) = 0$. To use the meta-material for the demonstration of the negative refraction phenomenon, it is preferable to have a broader frequency range of negative permeability. It is convenient to measure this broadness by the ratio

$$\Delta = \frac{\tilde{\omega}^2 - \omega_r^2}{\omega_r^2}, \tag{27}$$

which is, according to equation (23),

$$\Delta = \frac{\pi^2 \mu_0 r_0^4 n_0 L^{-1}}{1 + \mu_0 r_0 \Sigma L^{-1} - \dfrac{2}{3}\pi^2 \mu_0 r_0^4 n_0 L^{-1}}\cdot \tag{28}$$

The calculated dependency of this expression on the lattice constant b for different lattices is shown in Figure 3. The general tendency for all lattices is the significant broadening of the negative permeability range as the volume concentration of loops increases. It can be clearly seen that the hexagonal order within a plane provides considerably wider broadening than the square one. Shifting of neighbouring layers is very profitable at small values of b. As a result, we can conclude that the shifted hexagonal order is the most appropriate. It provides an up to three times broader frequency range compared with the tetragonal lattice used in [4].

To explain the importance of the lattice type for the value Δ it is useful to notice that with decrease in b both the lattice sum Σ and the concentration n_0 increase markedly. In the denominator of equation (28) the corresponding last two terms partially compensate each other. Therefore, the difference in the lattice sums for different lattices at a given density, which is not so crucial for the resonance position, plays an important role here. The smaller is Σ, the larger is Δ. The contribution to the sum coming from different loops may have the opposite signs: the largest positive terms arise from the loops just one above another in neighbouring layers, while contribution from the neighbouring loops within the same layer provide the largest negative terms. Therefore, Δ is smallest in the case of tetragonal lattice – there are only 4 neighbouring loops within the same layer, while the influence of the inter-layer loops is great. The hexagonal lattice provides 6 in-layer neighbours and also a higher loop concentration n_0. In the shifted lattices positive inter-layer terms are reduced, especially for the small b, when the strong influence of adjacent vertical neighbours (one exactly above another) is eliminated. This makes clear why the shifted hexagonal lattice gives the broadest negative permeability region.

4 Numerical simulation of the finite meta-structure

The circular conductive elements provide a rare opportunity to obtain the analytical expression for the effective response of the meta-material. Nevertheless, there is a number of interesting structural units which could be

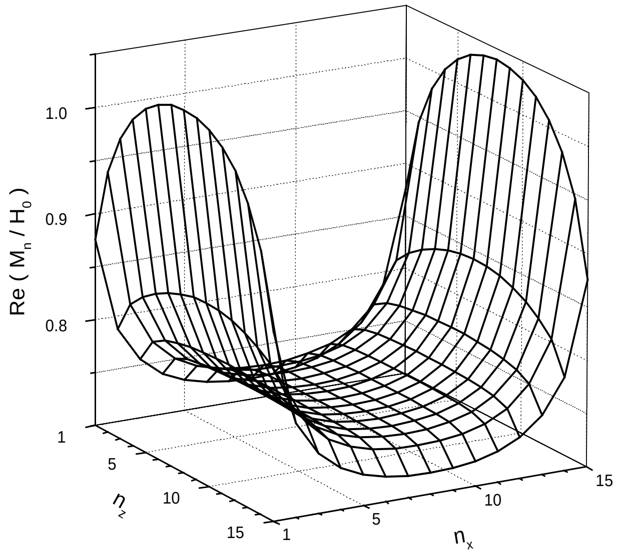

Fig. 4. Numerically calculated normalised magnetization distribution in the sample consisting of 15 × 15 × 15 loops.

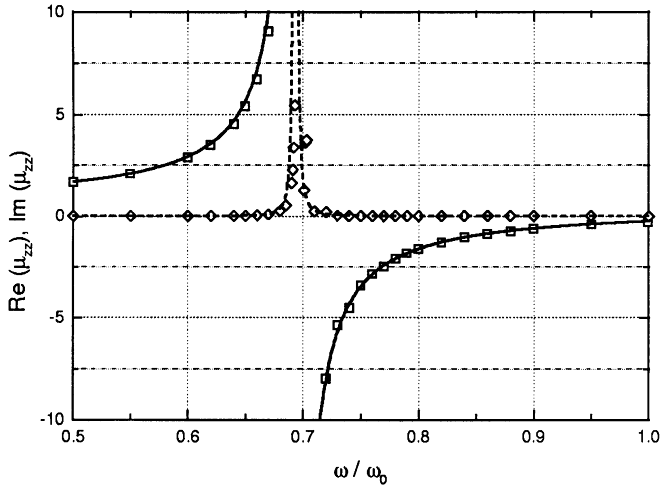

Fig. 5. Frequency dispersion of the permeability, obtained numerically (squares for the real and diamonds for the imaginary part), compared with the analytical formula (solid line, real part; dashed line, imaginary part). Calculations were made for the tetragonal lattice with $a = 2.1\,r_0$, $b = 0.5\,r_0$.

treated only numerically. In a numerical simulation of a meta-structure behaviour one can deal only with a finite sample and it is worthwhile from the computational point of view to minimize the number of structural units, yet keeping in mind that the macroscopic approach must be still valid. Below we determine numerically the macroscopic parameters of the loop meta-structure. This allows us to illustrate the postulates and principles of the macroscopic approach and to check the results obtained above. On the other hand, this comparison allows to optimise the numerical method itself, as one can determine the proper operating range of the method, the minimal sufficient size of the meta-structure, etc.

We study a rectangular sample of the meta-structure under the action of a homogeneous external magnetic field which acts along the z-direction, normal to the loop planes. The currents induced in different loops are now different and can be found with the help of the multi-impedance matrix equation (see Eqs. (1–2) for comparison):

$$\sum_{n'} Z_{nn'} I_{n'} = \mathrm{i}\omega\mu_0\pi r_0^2 H_{0z}. \tag{29}$$

where the diagonal terms of the impedance matrix are all the same and equal to the self-impedance (3), while the non-diagonal ones are given by equations (16–17). Calculating and inverting the $Z_{nn'}$ matrix we can get the currents in each loop numerically.

To demonstrate a typical current distribution inside the material we show the magnetization pattern inside the sample consisting of 15 × 15 × 15 loops (Fig. 4). Calculations were made for the tetragonal lattice with $a = 2.1\,r_0$ and $b = 0.5\,r_0$, *i.e.*, for the structure with remarkable mutual effects, at the frequency $0.6\,\omega_0$. The position of a loop is characterized by three numbers (n_x, n_y, n_z) taking values from 1 to 15. We show the distribution in the plane $n_y = 8$. The magnetization was determined for each loop in accordance with equation (4) with $I = I_n$, and normalized to H_0. The imaginary part was three orders of magnitude smaller than the real one and is not shown. It can be clearly seen that there is a noticeable flat region in the central area where the magnetization is almost homogeneous. Close to the borders, subsurface areas with specific properties are found. The characteristic thickness of these perturbed layers is about several lattice constants, confirming the idea of the local response formation within such a length.

The current distribution allows us to calculate the microscopic magnetic field according to equation (6). We assume the central unit cell in the sample to be the most representative for the bulk; then the averaging procedure (8) can be carried out by numerical integration over this unit cell. Using the value of the magnetization (4) in this cell, we get finally the permeability value according to equation (13). The effective permeability obtained in this way appears to be independent of the shape of the sample; the only necessary condition is that the sample dimensions should be larger than the response formation length. For instance, the results of the numerical simulation are the same for the samples of 11 × 11 × 31 and 19 × 19 × 11 loops. Actually, the accuracy of about one percent could be achieved already by using the sample having 11 loops in each dimension.

The comparison of the numerically obtained permeability as a function of frequency with the analytical formula (23) is shown in Figure 5. The current distribution remains similar to Figure 4 within the whole frequency range except for the two narrow regions $\omega = (0.700 \pm 0.002)\,\omega_0$ in the close vicinity of the resonance and $\omega = (1.2 \pm 0.2)\,\omega_0$ around the frequency at which the permeability is zero. Inside these regions a periodical modulation of the currents appears, with the period of the modulation being of the order of a few lattice constants. We believe that this inhomogeneity results from the excitation of standing exciton waves with extremely large wave vectors. Such waves are well known in optics [11]. Due to their small wavelength, they arise in theory even if retardation is neglected, being called Coulomb-type excitons. By analogy, in the magnetic case one should possibly call such waves Biot-Savart-type excitons.

The short-range inhomogeneity of the current distribution at frequencies mentioned above makes the response there non-local. Corresponding numerical values of the permeability are physically meaningless in these narrow frequency intervals. We omit them in the resulting curve in Figure 5. In all the remaining frequency range one can observe excellent agreement between analytical and numerical results for the real part of the permeability. The very stiff steep behaviour of the imaginary part near the resonance leads to the large sensitivity of the result to the errors of the numerical calculations. Nevertheless, we can report that the resonance frequency was determined numerically with an error of less than one percent.

This analysis allows us to consider such numerical calculations to be a very promising way to predict the meta-material response in the case of more complicated structural units.

5 Discussion

We demonstrated the capability of the macroscopic approach to describe successfully the properties of a meta-material constructed of circular conductive elements. The method of calculating the effective magnetic response can easily be generalized for the case of more complicated structural units and various types of their ordering in meta-materials. For instance, it allows generalization for the so-called 3D lattice where the planes of conductive elements are arranged in three perpendicular directions [10]. For the cases when the unit cell is too complicated for an analytical consideration, the suggested numerical method is nevertheless applicable. We have shown that for a very precise ab initio permeability calculation it is necessary to be capable of studying numerically a system of about thousand structural units.

We believe that further application of the ideas well developed in the optical theory would be extremely helpful in the new intensively growing field of meta-materials. It appears that a number of ideas and concepts, which have been approved in optics, are easily applicable to this case. For example, one can notice, that the structure of the expression (18) for the permeability has very much in common with the permittivity of uniaxial crystals, obtained in terms of the optical local field theory [9]. Moreover, taking the limit of low loop concentration and assuming the lattice to be cubic, one can show Σ to be equal to zero due to symmetry, which leads to the relation

$$\frac{\mu_{zz}-1}{\mu_{zz}+2}=\frac{1}{3}n_0\frac{-\mathrm{i}\omega\mu_0 r_0}{Z}\pi^2 r_0^3. \tag{30}$$

This shows a direct analogy to the Clausius-Mossotti relation for the permittivity of an isotropic substance with point polarizable particles. The product $-\mathrm{i}\omega\mu_0 r_0 Z^{-1}\pi^2 r_0^3$ plays here the role of the magnetic polarizability of a single structural unit.

The authors are grateful to Professor Laszlo Solymar for the fruitful discussions. Financial support from the Deutsche Forschungsgemeinschaft is acknowledged by E. S. (Emmy-Noether Program) and M. L. (Graduate College 695).

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